Sheaf Cohomology

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Given a short exact seq of sheaves on a scheme

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

The induced map $\Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'')$ need not be surjective.

Can we systematically measure the failure of this right exactness? Sheaf cohomology theory produces an answer. Indeed, we will attach functors $\{H^i(X,-)\}_{i\in\mathbb{N}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X(X)\text{-mod}$ such that the above exact seq will produce an exact seq

$$0 \to \underset{\Gamma(X,\mathcal{F}')}{\underset{\shortparallel}{H^0(X,\mathcal{F}')}} \to \underset{\Gamma(X,\mathcal{F})}{\underset{\shortparallel}{H^0(X,\mathcal{F})}} \to \underset{\Gamma(X,\mathcal{F}'')}{\underset{\shortparallel}{H^0(X,\mathcal{F}'')}} \to H^1(X,\mathcal{F}') \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{F}'')$$

$$\vdots$$

i.e The failure of surjectivity $\left(= \dfrac{H^0(X,\mathcal{F}'')}{\operatorname{Im}(X,\mathcal{F})}\right)$ is the kernel of the map $H^1(X,\mathcal{F}') \to H^1(X,\mathcal{F})$ and so on.

Instead of working with the right exactness of the special functor $\Gamma(X,-)$, we consider the same problem for more general functors and device an analogous solution.

Convention: Functors are between abelian categories, all functors are additive.

Eg: $\Gamma: \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_X(X)}$, For a map of ringed spaces or schemes $f: X \to Y$, $f_*: \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$; $f_*: \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$, For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)$.

§ Generalities:

Lemma. Effacable δ-functors are universal.

Pf. Let $\{T^i\}_{i\in\mathbb{N}}$ be an effacable δ-functor $\mathcal{A} \to \mathcal{B}$. Given another δ-functor $\{S^i\}_{i\in\mathbb{N}}$ and a natural transformation $f: T^0 \to S^0$, construct $f^n: T^n \to S^n$ s.t $f^0 = f$ by induction on n.

Set $f^0 = f$, Suppose $f^1, \ldots, f^n$ are already constructed.

Given $A \in \mathcal{A}$, fix an exact seq

$$0 \to A \to I \to I'' \to 0 \quad \text{s.t } F^{n+1}(A) \to F^{n+1}(I) \text{ is}$$

the zero map.

Have a diag

$$\begin{tikzcd}
\to T^n(A) \arrow[r] \arrow[d, "f^n(A)"] & T^n(I) \arrow[r] \arrow[d, "f^n_{(I)}"] & T^n(I'') \arrow[r] \arrow[d, "f^n_{(I'')}"] & T^{n+1}(A) \arrow[r, "0"] \arrow[d, dotted] & T^{n+1}(I) \\
\to S^n(A) \arrow[r] & S^n(I) \arrow[r] & S^n(I'') \arrow[r] & S^{n+1}(A) \arrow[r] & S^{n+1}(I)
\end{tikzcd}$$

$$\begin{tikzcd}
T^{n+1}(A) = \operatorname{coker}(T^n(I) \to T^n(I'')) \arrow[d] \arrow[ddr, bend right, "f^{n+1}(A)"'] & \\
\operatorname{coker}(S^n(I) \to S^n(I'')) = \operatorname{Ker}(S^{n+1}(A) \to S^{n+1}(I)) \arrow[d, hook] & \\
& S^{n+1}(A)
\end{tikzcd}$$

We need to check

(i) That $f^{n+1}(A)$ does not depend on the choice of $0 \to A \to I$

(ii) functoriality, given $A \to A'$, have

$$\begin{tikzcd}
T^{n+1}(A) \arrow[r, "f^{n+1}(A)"'] \arrow[d] & S^{n+1}(A) \arrow[d] \\
T^{n+1}(A') \arrow[r, "f^{n+1}(A')"'] & S^{n+1}(A')
\end{tikzcd}$$

commutative.

(i) Given $0 \to A \to I$, $\quad 0 \to A \to I'$

consider $0 \to A \to I \oplus I'$ (diag)

Have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "id", equal] & I \arrow[r] \arrow[d, shift right] & I/A \arrow[r] \arrow[d, shift right] & 0 \\
0 \arrow[r] & A \arrow[r] & I \oplus I' \arrow[r] \arrow[u, shift right] & I \oplus I'/A \arrow[r] \arrow[u, shift right] & 0
\end{tikzcd}$$

This gives

$$\begin{tikzcd}
T^n(A) \arrow[r] \arrow[d, "id"] & T^n(I) \arrow[r] \arrow[d] & T^n(I/A) \arrow[r] \arrow[d] & T^{n+1}(A) \arrow[d, "id"] \\
T^n(A) \arrow[r] & T^n(I \oplus I') \arrow[r] & T^n(I \oplus I'/A) \arrow[r] & T^{n+1}(A)
\end{tikzcd}$$

showing the extensions given by $0 \to A \to I$ and $0 \to A \to I \oplus I'$ are the same.

Now compare $0 \to A \to I'$ and $0 \to A \to I \oplus I'$

(iii) Given $0 \to A \xrightarrow{\alpha} I_A$, $0 \to B \xrightarrow{\beta} I_B$, and $g : A \to B$, replace $0 \to A \to I_A$ by

$0 \to A \xrightarrow{(\alpha,\, \beta \cdot g)} I_A \oplus I_B$, Then $T^{n+1}(A) \to T^{n+1}(I_A \oplus I_B) = T^{n+1}(I_A) \oplus T^{n+1}(I_B)$

$(T^{n+1}\alpha,\, T^{n+1}\beta \cdot T^{n+1}(g)) = 0$

Hence a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & I_A \oplus I_B \arrow[d, "pr_2"] \\ 0 \arrow[r] & B \arrow[r] & I_B \end{tikzcd}$$

which induces $f^{n+1}(g) : T^{n+1}(A) \to T^{n+1}(B)$

The commutativity of the left square below is clear. Since the biggest possible square of the diag below follows from the def of $f^{n+1}(A)$, $f^{n+1}(B)$ and the well definedness. The commutativity of the right square also follows

$$\begin{tikzcd} \mathrm{coker}(T^n(I_A \oplus I_B) \to T^n(A)) = T^{n+1}(A) \arrow[r, "f^{n+1}(A)"] \arrow[d, "\text{coming from } *"] \arrow[d, phantom, shift left=6ex, "\wr"] \arrow[d, shift left=14ex, "T^{n+1}(g)"] & \delta^{n+1}(A) \arrow[d, "\delta^{n+1}(g)"] \\ \mathrm{coker}(T^n(I_B) \to T^n(B)) = T^{n+1}(B) \arrow[r, "f^{n+1}(B)"'] & \delta^{n+1}(B) \end{tikzcd}$$

Thm. Let $\mathcal{A}$ be an abelian category with enough injectives, $F : \mathcal{A} \to \mathcal{B}$ additive, left exact, functor. Then there exists a unique universal δ functor $\{R^iF\}_{i \in \mathbb{N}}$ such that $R^0F = F$.

R^iF is called the i-th (right) derived functor of F

Pf. For $A \in \mathcal{A}$, choose an injective resolution $A \to I^\bullet$, define $R^iF(A) = H^i(F(I^\bullet))$

- Since any two injective resolutions are homotopic, different choices of resolutions $I^\bullet$ give isom $\{R^iF(A)\}$.

• Since any two injective resolutions are homotopic, different choices of resolutions $I^\bullet$ give isom $\{R^iF(A)\}_{i\in\mathbb{N}}$.

• $R^0F(A) = A$; F is left exact so
$$0 \to F(A) \to F(I^0) \to F(I^1) \text{ is exact}$$
$$\Rightarrow H^0(F(I^\bullet)) \xleftarrow{\sim} F(A).$$

• Universality: follows from the obs.

Prop. If $I \in \mathcal{A}$ is an injective object. Then $R^iF(I) = 0, \forall i > 0$.

pf. Take the injective resolution
$$I \to J^\bullet \quad \text{where } J^0 = I, \ J^i = 0 \ \forall i > 0. \ \square.$$

Rmk. We can play the same game for right exact functors when there are enough projectives.

End of 06.12.24 lecture

Def/ Notation: $(X, \mathcal{O}_X)$ ringed space, $R^i\Gamma(X, -) =: H^i(X, -)$

• For a topological space T, we can consider the category of sheaves of abelian groups and define $H^i(T, -)$
$$:= R^i\Gamma(-)$$

note a sheaf of abelian gps can be thought of as a sheaf of module over the sheaf of rings $\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the sheafification of the constant sheaf $U \to \mathbb{Z}$.

• $F = Hom_{\mathcal{O}_X}(\mathcal{F}, -)$ $\quad R^iF(-) =: Ext^i_{\mathcal{O}_X}(\mathcal{F}, -)$.

Def. $F: \mathcal{A} \to \mathcal{B}$ additive left exact. $J \in \mathcal{A}$ is called F acyclic if $R^iF(J) = 0 \ \forall i > 0$.

Prop. $F: \mathcal{A} \to \mathcal{B}$ (additive) left exact functor. For $A \in \mathcal{A}$, suppose there is a resolution of A by F acyclic objects, i.e $\exists$ complex
$$J^0 \to J^1 \to \cdots \quad \in \mathcal{A} \text{ such that each}$$
J^i is acyclic and $H^0(J^\bullet) \xleftarrow{\sim} A$ and $H^i(J^\bullet) = 0 \ \forall i > 0$.

Then $\quad R^iF(A) \xrightarrow{\sim} H^i(F(J^\bullet))$

Pf: Given an injective resolution $A \to I^\bullet$, we have a map of complexes have a map of complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal, "id"'] & I^0 \arrow[r] \arrow[d, dotted] & I^1 \arrow[r] \arrow[d, dotted] & \cdots \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

This induces a map $R^iF(A) = H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$

This map is an isom $\forall i$, but we don't verify that. Instead we show $R^iF(A) \xrightarrow{\sim} H^i(F(J^\bullet))$ abstractly.

- For $i=0$, $R^0F(A) = A \xrightarrow{\sim} H^0(F(J^\bullet)) \xrightarrow{\sim} A$ as F is left exact.
- We induce on i. Suppose we have isom $\forall A \in \mathcal{A}$, and $i \leq n$

Let $A' = \operatorname{coker}(A \to J^0)$, Then $J^\bullet$ gives a resolution $0 \to A' \to J^1 \to J^2 \to \cdots$

The exact seq $0 \to A \to J^0 \to A' \to 0$ gives

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(J^0) \arrow[r] \arrow[d, equal] & F(A') \arrow[r] \arrow[d, equal] & R^1F(A) \arrow[r] \arrow[d, "\wr"] & 0 \arrow[r] & R^1F(A') \arrow[r] & R^2F(A) \arrow[r] & 0 \\
& & F(J^0) \arrow[r] & \ker(F(J^1) \arrow[d] \arrow[r] & H^1(F(J^\bullet)) & & & & \\
& & & F(J^2)) & & & & &
\end{tikzcd}$$

$\Longrightarrow \quad R^1F(A) \simeq H^1(F(J^\bullet))$

and $\quad R^iF(A') \simeq R^{i+1}F(A) \quad \forall i \geq 1$

Since
$$\begin{tikzcd}
R^iF(A') \arrow[r, "\simeq"] \arrow[d, "\wr"] & H^i(J^\bullet[1]) \arrow[d, "\wr"] \\
R^{i+1}F(A) \arrow[r, "\simeq"] & H^{i+1}(J^\bullet)
\end{tikzcd}$$
for $i \leq n$

we are done. ∎

Def: A sheaf $\mathcal{F}$ on a topological space is called flasque if for any two open's in X U, V s.t $U \subseteq V$, $\mathcal{F}(V) \to \mathcal{F}(U)$ is surj.

Prop: $(X, \mathcal{O}_X)$ ringed space. Every injective $\mathcal{O}_X$ mod is flasque

Pf. For every open $U \subseteq X$, define $(j_U)_!(\mathcal{F}$

Pf. For every open $U \subseteq X$, define $(j_U)_!(\mathcal{F})$ to be the sheafification of

$$V \longmapsto 0 \quad \text{if } V \nsubseteq U$$
$$V \longmapsto \mathcal{F}(V) \quad \text{if } V \subseteq U$$

Given $V \subseteq U$ opens

Have an injection $0 \to (j_V)_! \mathcal{O}_X \to (j_U)_! \mathcal{O}_X$

Since $\mathcal{I}$ is injective, $\mathrm{Hom}(-, \mathcal{I})$ gives a surjection

$$0 \leftarrow \mathrm{Hom}_{\mathcal{O}_X}((j_V)_! \mathcal{O}_X, \;) \leftarrow \mathrm{Hom}_{\mathcal{O}_X}((j_U)_!, \;)$$
$$0 \leftarrow \mathcal{I}(V) \longleftarrow \mathcal{I}(U)$$

Prop: $(X, \mathcal{O}_X)$ ringed space. For an exact ses $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ ses in $\mathrm{Mod}_{\mathcal{O}_X}$

if $\mathcal{F}'$ is flasque.

Then $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ is exact

Pf. HW (Apply Zorn's lemma)